ROOT-POWERING OF POLYNOMIAL EQUATIONS

By

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PREFACE AND ACKNOWLEDGMENTS

The initial motivation for this work was a reference by Professor D. V. Thampuran, Mathematics Department, University of Florida, to a paper by A. C. Aitken [1] on root-cubing of polynomial equations. This led to the idea of root-powering in general and of the simplification of root determination from consecutively powered equations. That is, if equations could be derived, whose roots are consecutive powers of the roots of an original equation, the determination of those roots would be simplified.

Calculation of roots from consecutively powered equations is discussed in Chapter III of this paper. Several methods of general root-powering are derived in Chapter II, with emphasis on consecutive powers. Chapter I is devoted mainly to a brief discussion of the classical root-squaring method and some recent modifications of it.

Deep gratitude of this writer is owed the chairman of his supervisory committee, Professor Herbert A. Meyer, Department of Mathematics, University of Florida, for his generous giving of time, interest and advice. He has gone far beyond the requirements of professional duty to facilitate this work. Appreciation is due, also, the other members of the committee--Professor J. T. Moore, Professor W. P. Morse and Professor R. G. Selfridge, all of the Department of Mathematics, and Professor F. Zabeeh, Department of Philosophy, University of Florida--for their willingness to serve and their helpfulness whenever called upon. Finally, the generosity of The Danforth Foundation in providing financial support for a total of twenty months leading up to and through the writing of this dissertation has been a major factor in its accomplishment.

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CHAPTER I

THE ROOT-SQUARING METHOD

Historical background

A method for approximating all of the roots of a polynomial equation simultaneously by using a "root-squaring" process to separate the roots has been variously attributed to three men; Dandelin, Graeffe or Lobachevskii [11]. The first, G. Dandelin, in four appendices to a paper [7] published in 1826, described the basic procedure for "root-squaring" and indicated how it might be used. However, his work apparently received little notice and it remained for C. H. Graeffe [10] in 1837 to develop the method essentially as it is used today. In much of the subsequent literature on the subject, the method bears the latter's name; "Graeffe Root-Squaring." In 1834, Lobachevskii included a brief section in his book Algebra [13] on the root-squaring technique, and some authors, particularly Russians, are inclined to give him the credit. But, according to C. A. Hutchinson [12], the fundamental idea involved goes back to two papers by Edward Waring, "Miscellanea analytica," 1762 and "Meditationes analyticea," 1776.

The root-squaring algorithm

The root-squaring process itself is based on the very elementary product of a sum and a difference of like quantities.

The general polynomial of degree n with real coefficients,

$$P(x) = a_0x^n + a_1x^{n-1} + \cdot \cdot \cdot + a_{n-1}x + a_n, \tag{1.1}$$

written in factored form becomes

$$(1.2)\quad P(x) = a_o(x - x_1)(x - x_2) \cdot\cdot\cdot (x - x_n),$$

with $x_1, x_2, \cdot\cdot\cdot , x_n$ the n roots of $P(x) = 0$. Clearly, using (1.2),

$$\begin{aligned}(-1)^n P(-x) P(x) &= (-1)^n a_o^2 [(-x-x_1)(-x-x_2) \cdot\cdot\cdot (-x-x_n)] \\ &\qquad [(x-x_1)(x-x_2) \cdot\cdot\cdot (x-x_n)] \\ &= a_o^2 [(x+x_1)(x+x_2) \cdot\cdot\cdot (x+x_n)] [(x-x_1)(x-x_2) \\ &\qquad \cdot\cdot\cdot (x-x_n)] \\ &= a_o^2 (x^2-x_1^2)(x^2-x_2^2) \cdot\cdot\cdot (x^2-x_n^2) \\ (1.3)\qquad &= a_o^2 (y-x_1^2)(y-x_2^2) \cdot\cdot\cdot (y-x_n^2) = P_2(y),\end{aligned}$$

and $P_2(y) = 0$ is an equation whose roots are the squares of the roots of $P(x) = 0$.

The coefficients of

$$(1.4)\quad P_2(y) = A_o y^n + A_1 y^{n-1} + \cdot\cdot\cdot + A_{n-1} y + A_n$$

are compactly represented by the convolution product [9]*

$$(1.5)\quad \left\{B_k\right\} = \left\{(-1)^k a_k\right\} * \left\{a_k\right\}, \qquad A_k = B_{2k} \ (k > n,\ a_k = 0).$$

Either from this or directly from

$$(1.6)\quad (-1)^n P(-x) P(x) = A_o x^{2n} + A_1 x^{2(n-1)} + \cdot\cdot\cdot + A_{n-1} x^2 + A_n,$$

the A's are found to be

$$(1.7)\quad A_k = (-1)^k [a_k^2 + 2\sum_{j=1}^{k} (-1)^j a_{k-j} a_{k+j}].$$

---

*The convolution product $\left\{B_k\right\} = \left\{a_k\right\} * \left\{b_k\right\}$ defines a sequence $\left\{B_k\right\}$ in terms of two given sequences $\left\{a_k\right\}$ and $\left\{b_k\right\}$, $k = 0, 1, 2, \cdot\cdot\cdot$ by

$$B_k = a_o b_k + a_1 b_{k-1} + \cdot\cdot\cdot + a_{k-1} b_1 + a_k b_o.$$

The latter form is the basis for the usual organization of the work for a desk computer. n + 1 columns are headed by $a_o, a_1, \cdot \cdot \cdot, a_n$. Below each a_k is recorded the corresponding A_k', found by squaring the a_k directly above and totalling successively with alternating signs twice the products of the a_{k-j} and a_{k+j} symmetrically located on either side of a_k, until either a_o or a_n is reached. For a fourth degree equation, for example:

$$\begin{array}{ccccc} a_o & a_1 & a_2 & a_3 & a_4 \\ a_o^2 & a_1^2-2a_oa_2 & a_2^2-2a_1a_3+2a_oa_4 & a_3^2-2a_2a_4 & a_4^2 \end{array}$$

Note that the $(-1)^k$ in (1.7) is dropped, so the coefficients, A_k', correspond to an equation whose roots are the negatives of the squares of the roots of $P(x) = 0$, the so-called "Encke" roots. If this procedure is repeated on the A_k', etc. m times, the coefficients of an equation whose roots are the negative 2^m th powers of the roots of $P(x) = 0$ are obtained.

Root approximation

As suggested previously, the purpose of root-squaring is to further separate the roots of an algebraic equation. That is, if the moduli of the roots of $P(x) = 0$ are unequal, the moduli of high powers of these roots will differ much more. The squaring process is continued until the ratio of the modulus of any smaller root of the derived equation to that of any larger root is negligible compared to unity. Suppose the moduli are ordered,

$$|x_1| > |x_2| > \cdot \cdot \cdot > |x_{n-1}| > |x_n| ,$$

and that the above condition obtains at the $m^{\underline{th}}$ root-squaring. Letting $p = 2^m$, this implies

$$(1.8)\qquad \frac{\left|x^{p}_{j+r}\right|}{\left|x^{p}_{j}\right|} << 1\ ;\ j = 1,\ 2,\ 3,\ .\ .\ .\ ,\ n-1;\ r > 0.$$

If $P_p(x) = 0$, the equation whose roots are the $p^{\underline{th}}$ powers of $P(x) = 0$, is divided through by a_o, the resulting equation has the form

$$(1.9)\qquad x^n + B_1x^{n-1} + B_2x^{n-2} + \cdot\cdot\cdot + B_{n-1}x + B_n = 0.$$

Expressing these coefficients in terms of the elementary symmetric functions of the roots of $P_p(x) = 0$ and letting $x_j^p = X_j$, $j = 1, 2, \cdot\cdot\cdot , n$ gives [14, pp. 216-217]

$$(1.10)\qquad B_1 = -(X_1 + X_2 + \cdot\cdot\cdot + X_n) = -X_1\left(1 + \frac{X_2}{X_1} + \frac{X_3}{X_1} + \cdot\cdot\cdot + \frac{X_n}{X_1}\right)$$

$$B_2 = X_1X_2 + X_1X_3 + \cdot\cdot\cdot + X_1X_n + X_2X_3 + \cdot\cdot\cdot + X_2X_n + X_3X_4 + \cdot\cdot\cdot + X_{n-1}X_n$$

$$= X_1X_2\left(1 + \frac{X_3}{X_2} + \cdot\cdot\cdot + \frac{X_n}{X_2} + \frac{X_3}{X_1} + \cdot\cdot\cdot + \frac{X_n}{X_1} + \frac{X_3X_4}{X_1X_2} + \cdot\cdot\cdot + \frac{X_{n-1}X_n}{X_1X_2}\right)$$

$$B_3 = -(X_1X_2X_3 + X_1X_2X_4 + \cdot\cdot\cdot + X_1X_2X_n + \cdot\cdot\cdot + X_{n-2}X_{n-1}X_n)$$

$$= -X_1X_2X_3\left(1 + \frac{X_4}{X_3} + \cdot\cdot\cdot + \frac{X_n}{X_3} + \cdot\cdot\cdot + \frac{X_{n-2}X_{n-1}X_n}{X_1X_2X_3}\right)$$

$$\vdots$$

$$B_n = (-1)^nX_1X_2 \cdot\cdot\cdot X_n .$$

Using (1.8), these are approximated by

$$(1.11)\qquad B_1 \doteq -X_1,\ B_2 \doteq X_1X_2,\ B_3 \doteq -X_1X_2X_3$$

$$\cdot\cdot\cdot\ B_{n-1} \doteq (-1)^{n-1}X_1X_2 \cdot\cdot\cdot X_{n-1},\ B_n = (-1)^nX_1X_2 \cdot\cdot\cdot X_n.$$

Thus,

$$(1.12)\quad B_1 \doteq -X_1,\ \frac{B_2}{B_1} \doteq -X_2,\ \frac{B_3}{B_2} \doteq -X_3,\ \cdots,\ \frac{B_n}{B_{n-1}} \doteq -X_n$$

estimate the roots of $P_p(x) = 0$ and, hence, the $p^{\underline{th}}$ powers of the roots of $P(x) = 0$.

The usual criterion for the termination of the root-squaring process is for certain corresponding coefficients at one stage to increase, within the limits of precision required, by squaring at the next stage. That such "pivotal" coefficients exist, regardless of the number of roots with the same absolute value, is shown by Erwin H. Bareiss [2]. He also shows that ratios of consecutive pivotal coefficients give the moduli of all the roots.

The problems of complex roots, repeated roots, and roots with equal moduli are treated as special cases in most general works. C. A. Hutchinson [12] summarizes a number of the most common types in terms of the behavior of coefficients at successive stages of the root-squaring and the kind of roots implied. L. L. Cronvich [6] refines Hutchinson's list in detail and extends it to more cases. Once the moduli and types of roots are known, the signs of the real roots are easily found by substitution in the original equation. Determination of the complex roots is largely a matter of the ingenuity of the calculator--he may reduce the original equation with the determined real roots, utilize the symmetric functions of the roots or employ various transformations and other algebraic devices.

Method of Brodetsky and Smeal

Brodetsky and Smeal [3] avoid many of the difficulties of extracting high-order roots to obtain the real roots and of determining amplitudes of imaginary roots by applying the root-squaring procedure to $P(x + h)$ as well as to $P(x)$. Assuming h to be of small enough magnitude that higher than first-degree terms in h are negligible, they develop formulae for the coefficients of $P_2(x + h)$ in terms of those of $P_2(x)$, h and a set of b_i, $i = 0, 1, \cdots, n$. The latter, b_i, are, in turn, defined in terms of the coefficients of $P(x)$. As with the regular root-squaring process, this procedure can be iterated as many times as necessary. Finally, then, the real roots, with no ambiguity of sign, and the amplitudes of complex roots for a number of cases can be found from simple functions of ratios of the b's and the coefficients of $P_p(x)$.

Resultant procedure

Quite recently--1960--Erwin H. Bareiss [2] derived criteria for the number of root-squarings necessary to distinguish, as different, roots with moduli in a given ratio. He also found corresponding criteria for the acceptance or rejection of coefficients in $P_p(x)$ as pivotal; i.e., such that their ratios approximate the pth powers of the moduli of the roots of $P(x) = 0$. Once the set of pivotal coefficients and, hence, the set of differing moduli, is determined, the actual calculation of roots is accomplished by a "resultant procedure." For any given modulus, ρ, of any multiplicity for $P(x) = 0$, the resultant, $R(p)$, of $P(x)$ and $Q(x) = x^2 + px + \rho^2$ is calculated. In general, $R(p)$, is a polynomial of degree n in p. A necessary and sufficient condition for $P(x)$ and $Q(x)$ to have a common linear factor is for their resultant to vanish; thus, the solution of $R(p) = 0$ will yield all such factors. Since complex roots of $P(x) = 0$ always occur in conjugate pairs, α and $\bar{\alpha}$, if $x - \alpha$ is a common factor of $P(x)$ and $Q(x)$, so also is $x - \bar{\alpha}$. Then $Q(x) = (x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + \alpha\bar{\alpha}$ and $p = -(\alpha + \bar{\alpha})$ is real. Also, for roots of modulus ρ, only those roots of $R(p) = 0$ with $p^2 \leqq 4\rho^2$ need be considered. Hence, all and only those real roots of $R(p) = 0$ satisfying $p^2 \leqq 4\rho^2$, substituted into $Q(x) = 0$, will yield the roots of $P(x) = 0$ with modulus ρ.

Root-cubing

A. C. Aitken [1], in a brief note, describes a method of root-cubing; i.e., a method for finding $P_3(x) = 0$, the equation whose roots are the cubes of the roots of $P(x) = 0$. He shows how, in two stages, instead of the one stage of root-squaring, the coefficients of $P_3(x)$ can be determined. The calculations are similar to those for root-squaring in that they involve algebraic sums of binary products only; however, more terms are needed and the signs of the products are not simply alternating as in the latter process. He points out that the only significant advantage over root-squaring is that real roots are obtained without ambiguity of sign.

Aitken [16] has also extended the Bernoulli method for the largest root of an algebraic equation to all the roots. He employs functions related to the Newton symmetric functions and obtains the unpowered roots, but in a somewhat different form from that proposed in the following chapters.

CHAPTER II

GENERAL ROOT-POWERING

Introduction

In this chapter, four methods for raising the roots of a polynomial equation to any power are developed. The first utilizes a convolution product involving complex roots of unity. The second is based on a special property of circulant matrices. Ordinary characteristic-number-powering of matrices is adapted to the special case of polynomial equations for a third method, while in the fourth, it is pointed out how the relations developed by Newton between certain symmetric functions of the roots and the coefficients of P(x) can accomplish the same end.

The fundamental theorem

This theorem is based on the very elementary fact that the p distinct roots of the equation, $x^p - r^p = 0$, r a constant and p any positive integer, can be expressed in terms of r and p $p^{\underline{th}}$ roots of unity:

$$\left(\frac{x}{r}\right)^p = 1 , \quad x = \omega_p^k r, \quad k = 0, 1, 2, \cdot \cdot \cdot , p-1$$

where $\omega_p = e^{\frac{2\pi i}{p}}$. The basic theory of root-squaring and root-cubing discussed in Chapter I, as well as the theory of the first two methods of general root-powering below, derive more or less directly from the theorem.

Fundamental theorem: If

$$(2.1)\quad P(x) = a_o x^n + a_1 x^{n-1} + \ . \ . \ . + a_{n-1}x + a_n \ ; \ a_o, \ a_n \neq 0$$

$$= a_o(x-x_1)(x-x_2) \ . \ . \ . \ (x-x_n),$$

then

$$(2.2)\quad \prod_{j=0}^{p-1} P(\omega_p^j x^{\frac{1}{p}}) = (-1)^{n(p+1)} a_o^p \ (x-x_1^p)(x-x_2^p) \ . \ . \ . \ (x-x_n^p).$$

That is, an equation, $P_p(x) = 0$, whose roots are the $p^{\underline{th}}$ powers of the roots of $P(x) = 0$ is given by

$$\prod_{j=0}^{p-1} P(\omega_p^j x^{\frac{1}{p}}) = 0, \quad \text{where } \omega_p = e^{\frac{2\pi i}{p}} \ .$$

Proof:

$$\prod_{j=0}^{p-1} P(\omega_p^j x^{\frac{1}{p}}) = \prod_{j=0}^{p-1} a_o(\omega_p^j x^{\frac{1}{p}}-x_1)(\omega_p^j x^{\frac{1}{p}}-x_2) \ . \ . \ . \ (\omega_p^j x^{\frac{1}{p}}-x_n)$$

$$= \prod_{j=0}^{p-1} a_o \prod_{k=1}^{n} (\omega_p^j x^{\frac{1}{p}}-x_k)$$

$$= a_o^p \prod_{k=1}^{n} \prod_{j=0}^{p-1} (\omega_p^j x^{\frac{1}{p}}-x_k)$$

$$= a_o^p \prod_{k=1}^{n} \prod_{j=0}^{p-1} \left[-x^{\frac{1}{p}} \left(\frac{x_k}{x^{\frac{1}{p}}} - \omega_p^j \right) \right]$$

$$= a_o^p \prod_{k=1}^{n} (-1)^p \ x \prod_{j=0}^{p-1} \left(\frac{x_k}{x^{\frac{1}{p}}} - \omega_p^j \right)$$

$$= a_o^p \prod_{k=1}^{n} (-1)^p \ x \left[\left(\frac{x_k}{x^{\frac{1}{p}}} \right)^p - 1 \right]$$

$$= a_o^p \prod_{k=1}^{n} (-1)^p (x_k^p - x) = a_o^p \prod_{k=1}^{n} (-1)^{p+1}(x-x_k^p)$$

$$= (-1)^{n(p+1)} a_o^p (x-x_1^p)(x-x_2^p) \dots (x-x_n^p).$$

Corollary 1: If

$$(2.3) \quad (-1)^{n(p+1)} \prod_{j=0}^{p-1} P(\omega_p^j s) = A_o s^{np} + A_1 s^{np-1} + \dots + A_{np-1} s + A_{np},$$

then

$$(2.4) \quad \left\{A_k\right\} = \left\{a_k\right\} * \left\{\omega_p^k a_k\right\} * \left\{\omega_p^{2k} a_k\right\} * \dots * \left\{\omega_p^{(p-1)k} a_k\right\}.$$

That is, the sequence $\left\{A_k\right\}$ is generated by the p-fold convolution of sequences formed from the coefficients of $P(x)$ and powers of ω_p.

Proof:

$$\prod_{j=0}^{p-1} P(\omega_p^j s) = \prod_{h=1}^{p} P(\omega_p^{p-h} s), \quad j = p-h.$$

$$= \prod_{h=1}^{p} P(\omega_p^{-h} s) = \prod_{h=0}^{p-1} P(\omega_p^{-h} s),$$

since $\omega_p^p = 1$. For any particular h,

$$P(\omega_p^{-h} s) = a_o \omega_p^{-hn} s^n + a_1 \omega_p^{-h(n-1)} s^{n-1} + \dots + a_{n-1} \omega_p^{-h} + a_n$$

$$= \omega_p^{-hn} (a_o s^n + \omega_p^h a_1 s^{n-1} + \omega_p^{2h} a_2 s^{n-2} + \dots$$

$$+ \omega_p^{(n-1)h} a_{n-1} s + \omega_p^{nh} a_n)$$

$$= \omega_p^{-hn} p_h(s),$$

in which $p_h(s)$ can be considered as a generating function of the sequence $\left\{\omega_p^{hk} a_k\right\}$. Thus,

$$\prod_{j=0}^{p-1} P(\omega_p^j s) = \prod_{h=0}^{p-1} \omega^{-hn} p_h(s) = \prod_{h=0}^{p-1} \omega^{-hn} \prod_{h=0}^{p-1} p_h(s).$$

Now,

$$\prod_{h=0}^{p-1} \omega_p^{-hn} = \omega_p^{-n\sum_0^{p-1} h} = \omega_p^{-\frac{np(p-1)}{2}} .$$

If p is odd,

$$\omega^{-\frac{np(p-1)}{2}} = (\omega_p^p)^{-\frac{n(p-1)}{2}} = 1 ,$$

since $-\frac{n(p-1)}{2}$ is an integer and $\omega_p^p = 1$.

Also, if p is odd,

$$(-1)^{n(p+1)} = 1 = \prod_{h=0}^{p-1} \omega_p^{-hn} .$$

If p is even, $\omega_p^{\frac{p}{2}} = -1$, so that

$$\prod_{h=0}^{p-1} \omega_p^{-hn} = \left(\omega_p^{\frac{p}{2}}\right)^{-n(p-1)} = (-1)^{-n(p-1)} = (-1)^{n(p+1)}.$$

Thus, in either case,

$$\prod_{h=0}^{p-1} \omega_p^{-hn} = (-1)^{n(p+1)}$$

and

$$\prod_{j=0}^{p-1} P(\omega_p^j s) = \prod_{h=0}^{p-1} \omega^{-hn} \prod_{h=0}^{p-1} p_h(s) = (-1)^{n(p+1)} \prod_{h=0}^{p-1} p_h(s)$$

or

$$(-1)^{n(p+1)} \prod_{j=0}^{p-1} P(\omega_p^j s) = \prod_{h=0}^{p-1} p_h(s) .$$

A theorem on convolutions [9, p. 251] states that the product of the generating functions of two (or more) sequences gives the generating function of their convolution. Applying this theorem to the last equation above and noting that the right-hand side of (2.3) is just the generating function of $\left\{A_k\right\}$, we have the desired result, (2.4).

Corollary 2:

$$
(2.5)\qquad
\begin{aligned}
&A_1 = A_2 = \cdots = A_{p-1} = 0,\\
&A_{p+1} = A_{p+2} = \cdots = A_{2p-1} = 0,\\
&\vdots\\
&A_{(n-1)p+1} = A_{(n-1)p+2} = \cdots = A_{np-1} = 0,\\
&A_k = 0,\ k > np.
\end{aligned}
$$

Or, equivalently,

$$
(2.6)\quad (-1)^{n(p+1)} \prod_{j=0}^{p-1} P(\omega_p^j s) = \prod_{j=0}^{p-1} p_j(s) = A_0 s^{pn} + A_p s^{(n-1)p} + A_{2p} s^{(n-2)p} + \cdots + A_{np} .
$$

Proof: From the fundamental theorem, the right-hand side of (2.2) will involve only integral powers of x. Thus, (2.3) will involve only powers of s that are multiples of p, all other coefficients vanishing.

For an example, let n=4, p=3, so

$$P(x) = a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 .$$

Then,

$$
\begin{aligned}
P_3(x) &= P(x^{\frac{1}{3}})\ P(\omega_3 x^{\frac{1}{3}})\ P(\omega_3^2 x^{\frac{1}{3}})\\
&= A_0 x^4 + A_3 x^3 + A_6 x^2 + A_9 x + A_{12}
\end{aligned}
$$

and, dropping the subscript on ω,

$$\left\{A_k\right\} = \left\{a_k\right\} * \left\{\omega^k a_k\right\} * \left\{\omega^{2k} a_k\right\}.$$

Now, making use of

$$\omega^8 = \omega^5 = \omega^2,\quad \omega^7 = \omega^4 = \omega,\quad \omega^6 = \omega^3 = 1$$

and $\omega^2 + \omega = -1$, we have

$$\left\{\omega^k a_k\right\} = a_o,\ \omega a_1,\ \omega^2 a_2,\ a_3,\ \omega a_4$$

$$\left\{\omega^{2k} a_k\right\} = a_o,\ \omega^2 a_1,\ \omega a_2,\ a_3,\ \omega^2 a_4.$$

Letting $\left\{B_k\right\} = \left\{\omega^k a_k\right\} * \left\{\omega^{2k} a_k\right\}$ gives

k	B_k	a_k
o	a_o^2	a_o
1	$a_o a_1(\omega^2+\omega) = -a_o a_1$	a_1
2	$a_o a_2(\omega^2+\omega) + a_1^2 = a_1^2 - a_o a_2$	a_2
3	$2a_o a_3 + a_1 a_2\ (\omega+\omega^2) = 2a_o a_3 - a_1 a_2$	a_3
4	$a_o a_4(\omega^2+\omega) + a_1 a_3(\omega^2+\omega) + a_2^2 = a_2^2 - a_o a_4 - a_1 a_3$	a_4
5	$2a_1 a_4 + a_2 a_3(\omega^2+\omega) = 2a_1 a_4 - a_2 a_3$	
6	$a_2 a_4(\omega^2+\omega) + a_3^2 = a_3^2 - a_2 a_4$	
7	$a_3 a_4(\omega^2+\omega) = -a_3 a_4$	
8	a_4^2	

Finally, then, $\left\{A_k\right\} = \left\{a_k\right\} * \left\{B_k\right\}$ gives

k	A_k
0	a_o^3
1	$a_1 a_o^2 - a_1 a_o^2 = 0$
2	$a_o^2 a_2 + a_o a_1^2 - a_o^2 a_2 - a_o a_1^2 = 0$
3	$3a_o^2 a_3 - 3a_o a_1 a_2 + a_1^3$
4	$a_o a_2^2 - a_o^2 a_4 - a_o a_1 a_3 + a_o^2 a_4 + 2a_o a_1 a_3 - a_1^2 a_2 - a_o a_1 a_3 + a_1^2 a_2 - a_o a_2^2 = 0$
5	$2a_o a_1 a_4 - a_o a_2 a_3 - a_o a_1 a_4 + a_1 a_2^2 - a_o a_1 a_4 - a_1^2 a_3 + a_1^2 a_3 - a_o a_2 a_3 + 2a_o a_2 a_3 - a_1 a_2^2 = 0$
6	$3a_o a_3^2 - 3a_o a_2 a_4 + 3a_1^2 a_4 - 3a_1 a_2 a_3 + a_2^3$

7 $-a_0a_3a_4 + a_1a_3^2 - a_1a_2a_4 + 2a_1a_2a_4 - a_2^2a_3 + a_2^2a_3 - a_0a_3a_4$
$- a_1a_3^2 + 2a_0a_3a_4 - a_1a_2a_4 = 0$

8 $a_0a_4^2 - a_1a_3a_4 + a_2a_3^2 - a_2^2a_4 + 2a_1a_3a_4 - a_2a_3^2 + a_2^2a_4 - a_0a_4^2$
$- a_1a_3a_4 = 0$

9 $3a_1a_4^2 - 3a_2a_3a_4 + a_3^3$

10 $a_2a_4^2 - a_3^2a_4 + a_3^2a_4 - a_2a_4^2 = 0$

11 $a_3a_4^2 - a_3a_4^2 = 0$

12 a_4^3

Circulants

The equation whose roots are the $p^{\underline{th}}$ powers of the roots of $P(x) = 0$ can be obtained also by means of circulants. Form the p - square $(p > n)$ circulant $A_p(x) =$

$$(2.7)\quad \begin{vmatrix}
a_n & a_{n-1}x^{\frac{1}{p}} & a_{n-2}x^{\frac{2}{p}} & \cdots & a_1x^{\frac{n-1}{p}} & a_0x^{\frac{n}{p}} & 0 & \cdots & 0 & 0 \\
0 & a_n & a_{n-1}x^{\frac{1}{p}} & \cdots & a_2x^{\frac{n-2}{p}} & a_1x^{\frac{n-1}{p}} & a_0x^{\frac{n}{p}} & \cdots & 0 & 0 \\
0 & 0 & a_n & \cdots & a_3x^{\frac{n-3}{p}} & a_2x^{\frac{n-2}{p}} & a_1x^{\frac{n-1}{p}} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_0x^{\frac{n}{p}} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_1x^{\frac{n-1}{p}} & a_0x^{\frac{n}{p}} \\
a_0x^{\frac{n}{p}} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_2x^{\frac{n-2}{p}} & a_1x^{\frac{n-1}{p}} \\
a_1x^{\frac{n-1}{p}} & a_0x^{\frac{n}{p}} & 0 & \cdots & 0 & 0 & 0 & \cdots & a_3x^{\frac{n-3}{p}} & a_2x^{\frac{n-2}{p}}
\end{vmatrix}$$

$$\begin{vmatrix} \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n-3}x^{\frac{3}{p}} & a_{n-4}x^{\frac{4}{p}} & a_{n-5}x^{\frac{5}{p}} & \cdots & 0 & 0 & 0 & \cdots & a_{n-1}x^{\frac{1}{p}} & a_{n-2}x^{\frac{2}{p}} \\ a_{n-2}x^{\frac{2}{p}} & a_{n-3}x^{\frac{3}{p}} & a_{n-4}x^{\frac{4}{p}} & \cdots & 0 & 0 & 0 & \cdots & a_n & a_{n-1}x^{\frac{1}{p}} \\ a_{n-1}x^{\frac{1}{p}} & a_{n-2}x^{\frac{2}{p}} & a_{n-3}x^{\frac{3}{p}} & \cdots & a_o x^{\frac{n}{p}} & 0 & 0 & \cdots & 0 & a_n \end{vmatrix}$$

(2.7) Then, the determinant of A_p has the value [4, pp. 74-76]

$$\left| A_p \right| = \prod_{t=1}^{p} \left\{ a_n + a_{n-1}x^{\frac{1}{p}}\omega_t + a_{n-2}x^{\frac{2}{p}}\omega_t^2 + \cdots + a_1 x^{\frac{n-1}{p}}\omega_t^{n-1} + a_o x^{\frac{n}{p}}\omega_t^n \right\}$$

$$= \prod_{t=1}^{p} \left[a_o\left(\omega_t x^{\frac{1}{p}}\right)^n + a_1\left(\omega_t x^{\frac{1}{p}}\right)^{n-1} + \cdots + a_{n-1}\left(\omega_t x^{\frac{1}{p}}\right) + a_n \right]$$

$$= \prod_{t=1}^{p} P\left(\omega_t x^{\frac{1}{p}}\right),$$ where ω_t, $t = 1, 2, \ldots, p$ are the p distinct pth roots of unity. Thus, $\left| A_p(x) \right| = 0$ represents the desired nth degree equation by the fundamental theorem.

Note that A_p consists essentially of two parts: an upper diagonal part consisting of n+1 lines of identical elements parallel to the main diagonal of a_n's, and an n-square lower triangular part. We form the new matrix, $B_p(x)$, from $A_p(x)$ by multiplying the rows of the latter successively by $x^{\frac{r-1}{p}}$, where r is the number of the row, and factoring the same quantity from the rth column. This removes x completely from the upper diagonal part and reduces each of its exponents in the lower triangular part to unity. That is,

$$(2.8)\quad B_p(x) = \begin{pmatrix}
a_n & a_{n-1} & a_{n-2} & \cdots & a_1 & a_o & 0 & \cdots & & \\
0 & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_o & \cdots & & \\
0 & 0 & a_n & \cdots & a_3 & a_2 & a_1 & \cdots & & \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
 & & & \cdots & & & & \cdots & a_o & 0 \\
0 & & & \cdots & & & & \cdots & a_1 & a_o \\
a_ox & 0 & & \cdots & & & & \cdots & a_2 & a_1 \\
a_1x & a_ox & 0 & \cdots & & & & \cdots & a_3 & a_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
a_{n-3}x & a_{n-4}x & a_{n-5}x & \cdots & & & & \cdots & a_{n-1} & a_{n-2} \\
a_{n-2}x & a_{n-3}x & a_{n-4}x & \cdots & 0 & & & \cdots & a_n & a_{n-1} \\
a_{n-1}x & a_{n-2}x & a_{n-3}x & \cdots & a_ox & 0 & & \cdots & 0 & a_n
\end{pmatrix}.$$

For example, with $n = 3$, $p = 8$,

$$A_8(x) = \begin{pmatrix}
a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 & 0 & 0 & 0 \\
0 & a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 & 0 & 0 \\
0 & 0 & a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 & 0 \\
0 & 0 & 0 & a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 \\
0 & 0 & 0 & 0 & a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} \\
a_ox^{\frac{3}{8}} & 0 & 0 & 0 & 0 & a_3 & a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} \\
a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 & 0 & 0 & 0 & a_3 & a_2x^{\frac{1}{8}} \\
a_2x^{\frac{1}{8}} & a_1x^{\frac{2}{8}} & a_ox^{\frac{3}{8}} & 0 & 0 & 0 & 0 & a_3
\end{pmatrix}$$

and

$$B_8(x) = \begin{pmatrix} a_3 & a_2 & a_1 & a_o & 0 & 0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_o & 0 & 0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_o & 0 & 0 \\ 0 & 0 & 0 & a_3 & a_2 & a_1 & a_o & 0 \\ 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 & a_o \\ a_ox & 0 & 0 & 0 & 0 & a_3 & a_2 & a_1 \\ a_1x & a_ox & 0 & 0 & 0 & 0 & a_3 & a_2 \\ a_2x & a_1x & a_ox & 0 & 0 & 0 & 0 & a_3 \end{pmatrix}.$$

Clearly, because of the nature of the transformations on A_p, $|A_p| = |B_p|$; and $|B_p(x)| = 0$, also, represents an equation whose roots are the p^{th} powers of the roots of $P(x) = 0$.

By interchanging appropriate columns of the transpose of $B_p(x)$, we obtain the new matrix,

$$(2.9) \quad C_p(x) = \begin{pmatrix} a_ox & a_1x & a_2x & \cdots & a_{n-1}x & a_n & 0 & \cdots & & & & \cdots \\ 0 & a_ox & a_1x & \cdots & a_{n-2}x & a_{n-1} & a_n & 0 \cdots & & & & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & & \cdots & & & & \cdots & a_o & a_1 & a_2 \cdots & a_n \\ a_n & 0 & & \cdots & & & & \cdots & 0 & a_o & a_1 \cdots & a_{n-1} \\ a_{n-1} & a_n & 0 & \cdots & & & & \cdots & & 0 & a_o \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_2 & a_3 & a_4 & \cdots & 0 & & & \cdots & & & \cdots & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_n & 0 & & \cdots & & & \cdots & a_o \end{pmatrix},$$

a form in which the coefficients of $P(x)$ occur in their natural order in rows and x occurs only in the upper triangular part of the n-square upper left section.

The preceding example - $n = 3$, $p = 8$ - becomes

$$C_8(x) = \begin{pmatrix} a_0x & a_1x & a_2x & a_3 & 0 & 0 & 0 & 0 \\ 0 & a_0x & a_1x & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_0x & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ a_3 & 0 & 0 & 0 & 0 & a_0 & a_1 & a_2 \\ a_2 & a_3 & 0 & 0 & 0 & 0 & a_0 & a_1 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 & 0 & a_0 \end{pmatrix} .$$

Clearly, here

$$P_p(x) = \left| C_p(x) \right| , \tag{2.10}$$

and this is the form to be used in the sequel. We employ a Laplace expansion of $\left| C_p \right|$ on the first n rows and consider only the cases of $p \geqq 2n$; for $n < p < 2\,n$, $\left| C_p \right|$ would, in general, be of low enough order that any convenient method of expansion could be used.

With this condition, $p \geqq 2n$, C_p is partitioned

$$C_p = \begin{pmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{pmatrix} \tag{2.11}$$

where C'_{11} is n by 2n. If $p = 2n$, C'_{11} is just the first n rows of C_p, and C'_{12} and C'_{22} are omitted. For $p > 2n$, C'_{12} will have only zero elements. Thus, in a Laplace expansion on the first n rows of C_p, the set of non-zero minors on these n rows is invariant for all $p \geqq 2n$ and these minors are $\binom{2n}{n}$ in number. Hence, the expansion consists of the sum of $\binom{2n}{n}$ binary products of an n-square and a (p-n)-square determinant in which only the algebraic complements of the minors of C'_{11} vary with different p.

For $p \geqq 2n$, the product of a minor from C'_{11} and its algebraic complement from C'_{21} and C'_{22} can be characterized by a sequence of the integers 1 through 2n; the first n of them indicating the columns of C'_{11} chosen for its minor, the last n the columns of C'_{21} in the algebraic complement; C'_{22} constitutes the last p - 2n columns of every algebraic complement. The sign of the product is determined from the number of inversions in this sequence, since this will correspond in class to the number of transpositions of columns necessary to put the minor of C'_{11} in leading position of C_p. Note that permutations within the first or last n terms of any given sequence yield the same term of the expansion of $|C_p|$. To illustrate for n = 3, p = 8:

$$P_8(x) = \left| C_8(x) \right| = \begin{vmatrix} C'_{11} & C'_{12} \\ C'_{21} & C'_{22} \end{vmatrix} =$$

$$\begin{vmatrix} \begin{pmatrix} a_0x & a_1x & a_2x & a_3 & 0 & 0 \\ 0 & a_0x & a_1x & a_2 & a_3 & 0 \\ 0 & 0 & a_0x & a_1 & a_2 & a_3 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_0 & a_1 \\ a_3 & 0 & 0 & 0 & 0 & a_0 \\ a_2 & a_3 & 0 & 0 & 0 & 0 \\ a_1 & a_2 & a_3 & 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} a_3 & 0 \\ a_2 & a_3 \\ a_1 & a_2 \\ a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \end{vmatrix},$$

and, for example,

$$\left|2, 4, 5; 1, 3, 6\right|_8 = -\begin{vmatrix} a_1x & a_3 & 0 \\ a_ox & a_2 & a_3 \\ 0 & a_1 & a_2 \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ a_3 & 0 & a_o & a_1 & a_2 \\ a_2 & 0 & 0 & a_o & a_1 \\ a_1 & a_3 & 0 & 0 & a_o \end{vmatrix} ;$$

negative, since 2, 4, 5, 1, 3, 6, shows five, an odd number, of inversions. Similarly,

$$\left|1, 3, 6; 2, 4, 5\right|_8 = +\begin{vmatrix} a_ox & a_2x & 0 \\ 0 & a_1x & 0 \\ 0 & a_ox & a_3 \end{vmatrix} \cdot \begin{vmatrix} 0 & a_o & a_1 & a_3 & 0 \\ 0 & 0 & a_o & a_2 & a_3 \\ 0 & 0 & 0 & a_1 & a_2 \\ a_3 & 0 & 0 & a_o & a_1 \\ a_2 & 0 & 0 & 0 & a_o \end{vmatrix}$$

and

$$\left|2, 3, 4; 1, 5, 6\right|_8 = -\begin{vmatrix} a_1x & a_2x & a_3 \\ a_ox & a_1x & a_2 \\ 0 & a_ox & a_1 \end{vmatrix} \cdot \begin{vmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_o & a_1 & a_2 & a_3 \\ a_3 & 0 & a_o & a_1 & a_2 \\ a_2 & 0 & 0 & a_o & a_1 \\ a_1 & 0 & 0 & 0 & a_o \end{vmatrix} .$$

Now, consider the matrices C_p for any two consecutive values of p, say, $r \geqq 2n$ and $r + 1$. As established above, C'_{11} is invariant in C_r and C_{r+1}. C'_{21} adds a row of zeros at the top of the first n columns and at the bottom of the last n columns in going from C_r to C_{r+1}, while C'_{22} is augmented by a row of zeros at the bottom and a new column, beginning with a_o at the bottom. The minors from $(C'_{21}\ C'_{22})$ occurring in the expansion of $\left|C_{r+1}\right|$ can, in turn, be expanded on elements of their first or last rows; thus, we compare $(C'_{21}\ C'_{22})$ in C_r and C_{r+1} as follows:

I. For expansions on first-row elements.

  1. The first row of $(C'_{21}\ C'_{22})$ in C_{r+1} has all zeros except in columns n+1, n+2, . . . , 2n and 2n+1. The subscript on the a in each of these non-zero positions is n+1 less than the number of the column.

  2. If the first row of $(C'_{21}\ C'_{22})$ in C_{r+1} is deleted, the remaining matrix has all zeros in the $(n+1)^{\underline{th}}$ column; if this column of zeros is removed and the following columns each moved one to the left, the resulting matrix is identical to $(C'_{21}\ C'_{22})$ in C_r.

II. For expansion on last-row elements.

  1. The last row of $(C'_{21}\ C'_{22})$ in C_{r+1} consists of zeros except for the last element which is a_o, and the first n elements where the subscript on the a corresponds to the number of the column.

  2. If the last row of $(C'_{21}\ C'_{22})$ in C_{r+1} is deleted, the remaining matrix has all zeros in the $n^{\underline{th}}$ column; if this column of zeros is removed, the first n-1 columns moved one to the right and the last column placed in number one position, the resulting matrix is identical to $(C'_{21}\ C'_{22})$ in C_r.

The above observations, I and II, can be used to derive the form of recurrence rules for the expansion of $\left|C_p(x)\right|$ for $p > 2n$. Consider any particular sequence designating one of the binary determinant products in $\left|C_{r+1}\right|$, say

(2.12)

$$\left|N_{j+1},N_{j+2},\ . \ . \ .\ ,N_n,M_{k+1},M_{k+2},\ . \ . \ .\ ,M_n;N_1,N_2,\ . \ . \ .\ ,N_j,M_1,M_2,\ . \ . \ .\ ,M_k\right|_{r+1}$$

where $N_i \leqq n$, $n < M_i \leqq 2n$ and $j + k = n$. Then, representing the minors from $(C'_{21}\ C'_{22})$ by just the last n terms of our sequences and assuming the N's and M's in increasing order, we expand $\left|;N_1,\ . \ . \ .\ ,N_j,\ M_1,\ . \ . \ .\ ,M_k\right|_{r+1}$ on its first row, using I.

If $M_1 = n+1$,

$$\left|;N_1,\ . \ . \ .\ ,N_j,n+1,M_2,\ . \ . \ .\ ,M_k\right|_{r+1}$$

(2.13)

$$= (-1)^j a_0\left|;N_1,\ . \ . \ .\ ,N_j,M_2-1,\ . \ . \ .\ ,M_k-1,\ 2n\right|_r.$$

If $M_1 \neq n+1$,

$$\left|;N_1,\ . \ . \ .\ ,N_j,M_1,M_2,\ . \ . \ .\ ,M_k\right|_{r+1}$$

(2.14)

$$\begin{aligned} = (-1)^j[a_s&\left|;N_1,\ . \ . \ .\ ,N_j,\ M_2-1,M_3-1,\ . \ . \ .\ ,M_k-1,2n\right|_r \\ -a_t&\left|;N_1,\ . \ . \ .\ ,N_j,\ M_1-1,M_3-1,\ . \ . \ .\ ,M_k-1,2n\right|_r \\ &+\ . \ . \ . \\ +(-1)^{k-1}a_z&\left|;N_1,\ . \ . \ .\ ,N_j,M_1-1,M_2-1,M_3-1,\ . \ . \ .\ ,M_{k-1}-1,\ 2n\right|_r \\ +(-1)^k a_n&\left|;N_1,\ . \ . \ .\ ,N_j,\ M_1-1,M_2-1,\ . \ . \ .\ ,M_k-1\right|_r], \end{aligned}$$

where $s = M_1-n-1$, $t = M_2-n-1$, . . . , $z = M_k-n-1$.

This reduction by elements of the first row is summarized by the following rules:

1. If n+1 appears in the half-sequence, the reduction consists of a single term based on $M_i = n+1$; otherwise, each integer, M_i, greater than n yields a term.
2. The sign of a term is + or - according as the M_i is in odd or even position.
3. The subscript for a is $M_i-(n+1)$.
4. The determinant half-sequence for a term based on M_i is obtained by repeating the integers less than n+1, deleting the M_i, reducing each other integer greater than n by one and appending 2n at the end.
5. If n+1 does not appear, an additional term with factor $(-1)^n a_n$ and determinant half-sequence having integers greater than n reduced by one is included.

These rules are applied to the preceding examples (n=3):

$$\left| ; 1, 3, 6 \right|_8 = a_2 \left| ; 1, 3, 6 \right|_7 - a_3 \left| ; 1, 3, 5 \right|_7$$

$$\left| ; 2, 4, 5 \right|_8 = -a_0 \left| ; 2, 4, 6 \right|_7$$

$$\left| ; 1, 5, 6 \right|_8 = -a_1 \left| ; 1, 5, 6 \right|_7 + a_2 \left| ; 1, 4, 6 \right|_7 - a_3 \left| ; 1, 4, 5 \right|_7$$

Rules for the expansion of $\left| ; N_1, \ldots, N_j, M_1, \ldots, M_k \right|_{r+1}$ by elements of the last row are deduced similarly from observations II. First, we think of moving the last column of the determinant into leading position and the last row into top position, leaving the order of the remaining rows and columns unchanged. Since this would involve an even number of transpositions in total, the resulting determinant would be equivalent to $\left| ; N_1, \ldots, N_j, M_1, \ldots, M_k \right|_{r+1}$.

If $N_j = n$,

$$\begin{aligned} &\Big| \,; N_1, N_2, \ . \ . \ . \ , N_{j-1}, n, M_1, \ . \ . \ . \ , M_k \Big|_{r+1} \\ &= (-1)^j a_n \Big| \,; 1, N_1+1, N_2+1, \ . \ . \ . \ , N_{j-1}+1, M_1, \ . \ . \ . \ , M_k \Big|_r \ . \end{aligned} \tag{2.15}$$

If $N_j \neq n$, expanding on the new top row gives

$$\begin{aligned} &\Big| \,; N_1, N_2, \ . \ . \ . \ , N_j, M_1, \ . \ . \ . \ , M_k \Big|_{r+1} \\ &= a_o \Big| \,; N_1+1, N_2+1, \ . \ . \ . \ , N_j+1, M_1, \ . \ . \ . \ , M_k \Big|_r \\ &- a_s \Big| \,; 1, N_2+1, \ . \ . \ . \ , N_j+1, M_1, \ . \ . \ . \ , M_k \Big|_r \\ &+ a_t \Big| \,; 1, N_1+1, N_3+1, \ . \ . \ . \ , N_{j+1}, M_1, \ . \ . \ . \ , M_k \Big|_r \\ &- \ . \ . \ . \\ &+ (-1)^j a_z \Big| \,; 1, N_1+1, N_2+1, \ . \ . \ . \ , N_{j-1}+1, M_1, \ . \ . \ . \ , M_k \Big|_r \ , \end{aligned} \tag{2.16}$$

where $s = N_1$, $t = N_2$, . . . , $z = N_j$. This reduction by elements of the last row is summarized by the following rules:

1. If n appears in the half-sequence, the reduction consists of a single term based on $N_i = n$; otherwise, each integer, N_i, less than n yields a term.
2. The sign of a term is + or - according as the N_i is in even or odd position.
3. The subscript for a is N_i.
4. The determinant half-sequence for the term based on N_i is obtained by repeating the integers greater than n, deleting the N_i, increasing each other integer less than n by one and appending 1 at the beginning.

5. If n does not appear, an additional term with factor a_o and determinant half-sequence having integers less than n increased by one is included.

These rules are applied to the preceding examples ($n = 3$):

$$\left|\,;\,1,\,3,\,6\right|_8 = +\,a_3\left|\,;\,1,\,2,\,6\right|_7$$

$$\left|\,;\,2,\,4,\,5\right|_8 = a_o\left|\,;\,3,\,4,\,5\right|_7 - a_2\left|\,;\,1,\,4,\,5\right|_7$$

$$\left|\,;\,1,\,5,\,6\right|_8 = a_o\left|\,;\,2,\,5,\,6\right|_7 - a_1\left|\,;\,1,\,5,\,6\right|_7 \,.$$

The most economical procedure would be a combination of the two sets of rules:

1. If n or n+1 appears in the half-sequence, use the rule giving only a single term.
2. If half or fewer of the integers in the half-sequence are less than n, expand on the bottom row (last set of rules); otherwise, expand on the top row.

If the determinants of the minors from $(C'_{21}\ C'_{22})$ for $p = 2n$ are evaluated and coded by their corresponding half-sequences, each higher order determinant can be evaluated successively by the above rules in terms of the next lower order. It is observed that the number of integers greater than n in any half-sequence corresponds to the power of x associated with it; also, that the same power of x is associated with every half-sequence in its reduction by the above rules. Thus, the coefficients of $P_p(x)$ can be calculated independently of each other.

Matrix-powering

Without loss in generality for our purposes, we may assume the coefficient, a_o, of the highest power of x in P(x) to be one. Thus, (2.17) $P(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n$.

The companion matrix of (2.17) can be written

$$(2.18) \qquad A = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

Then,

$$(2.19) \qquad \left| A - \lambda I \right| = (-1)^n P(\lambda) ,$$

and the characteristic numbers of A are just the roots of $P(x) = 0$. It is also known that the characteristic numbers of any positive integral power of A are the same power of the characteristic numbers of A. Hence, powering the companion matrix, A, can provide a method of powering the roots of $P(x) = 0$.

Clearly, successive squarings of A, A^2, A^4, etc. would be analogous to the polynomial root-squaring discussed in Chapter I. However, to be useful, some criterion for the termination of the matrix-squaring process and a subsequent method for calculation of the separated characteristic numbers would need to be derived; this writer has been unable to do so.

Powering of A might be used to obtain equations whose roots are consecutive powers of the roots of $P(x) = 0$ and, thence, simplify the calculation of the original roots as discussed in Chapter III. That is, the usual polynomial root-squaring could be carried out to the power, say, $p = 2^m$, necessary to obtain the desired precision. Then, the matrix, A, could be powered by any convenient method to obtain A^p, A^{p+1}, etc. These consecutive powers of A are particularly easy to calculate because of the special form of A. Suppose

$$(2.20) \quad A^p = \begin{pmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{21} & b_{22} \cdots & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} \cdots & b_{nn} \end{pmatrix} .$$

Multiplication of A^p on the left by A, (2.18), shows that each of the rows of A^p is shifted down one and the last row is deleted. By multiplication of A^p on the right by (2.18) the new first row is easily obtained as indicated:

$$(2.21) \quad A^{p+1} = \begin{pmatrix} -a_1 b_{11}+b_{12} & -a_2 b_{11}+b_{13} & \cdots & -a_j b_{11}+b_{1,j+1} & \cdots & -a_{n-1} b_{11}+b_{1n} & -a_n b_{11} \\ b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1,n-1} & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2,n-1} & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & & b_{n-1,j} & & b_{n-1,n-1} & b_{n-1,n} \end{pmatrix} .$$

Finally, the characteristic equations for A^{p+1}, A^{p+2}, etc. are the desired equations.

Newton symmetric functions

A final method to obtain equations whose roots are consecutive powers of the roots of $P(x) = 0$ that might be used in conjunction with polynomial root-squaring can be based on the relations between the Newton symmetric functions and the coefficients of $P(x)$ in the form (2.17).

These relations can be expressed either in determinant form [5, p. 86] or in terms of a sequence of equations [15, p. 261] from which the determinants are easily derived.

Let

$$S_k = \sum_{j=1}^{n} x_j^k , \tag{2.22}$$

where the x_j are the roots of $P(x) = 0$.

Then,

$$S_1 = -a_1 \quad , \; S_2 = \begin{vmatrix} a_1 & 1 \\ 2a_2 & a \end{vmatrix} , \quad S_3 = - \begin{vmatrix} a_1 & 1 & 0 \\ 2a_2 & a_1 & 1 \\ 3a_3 & a_2 & a_1 \end{vmatrix} ,$$

or, in general,

$$S_k = (-1)^k \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ 2a_2 & a_1 & 1 & 0 & \cdots & 0 \\ 3a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ ka_k & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_1 \end{vmatrix} \tag{2.23}$$

for $k = 1, 2, 3, \ldots$, and $a_j = 0$ for $j > n$.

The coefficients, a_k, can, in turn, be expressed in terms of the S_j:

$$a_1 = - S_1 , \quad a_2 = \tfrac{1}{2} \begin{vmatrix} S_1 & 1 \\ S_2 & S_1 \end{vmatrix} , \quad a_3 = - \frac{1}{3!} \begin{vmatrix} S_1 & 1 & 0 \\ S_2 & S_1 & 1 \\ S_3 & S_2 & S_1 \end{vmatrix} ,$$

or, in general,

$$(2.24)\quad a_k = \frac{(-1)^k}{k!}\begin{vmatrix} S_1 & 1 & 0 & 0 & \cdots & 0 \\ S_2 & S_1 & 1 & 0 & \cdots & 0 \\ S_3 & S_2 & S_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ S_k & S_{k-1} & S_{k-2} & S_{k-3} & \cdots & S_1 \end{vmatrix}$$

for $k = 1, 2, \ldots, n$; $a_k = 0$ for $k > n$. The sequence of equations from which these determinants are derived is

$$(2.25)\quad \begin{aligned} &S_1 + a_1 = 0 \\ &S_2 + a_1S_1 + 2a_2 = 0 \\ &S_3 + a_1S_2 + a_2S_1 + 3a_3 = 0 \\ &\vdots \\ &S_k + a_1S_{k-1} + a_2S_{k-2} + \cdots + a_{k-1}S_1 + ka_k = 0 , \end{aligned}$$

where $a_j = 0$ for $j > n$.

Now, suppose the equation whose roots are the $p^{\underline{th}}$ powers of the roots of $P(x) = 0$ is written in the form

$$(2.26)\quad P_p(x) = x^n + A_1x^{n-1} + A_2x^{n-2} + \cdots + A_n = 0 .$$

That is, the roots of (2.26) are $x_1^p, x_2^p, \ldots, x_n^p$.

Then, clearly,

$$(2.27)\quad A_1 = \sum_{j=1}^{n} x_j^p = - S_p , \; A_2 = \frac{1}{2}\begin{vmatrix} S_1 & 1 \\ S_{2p} & S_p \end{vmatrix} , \; \ldots$$

$$A_n = \frac{(-1)^n}{n!} \begin{vmatrix} S_p & 1 & \cdots & 0 \\ S_{2p} & S_p & \cdots & 0 \\ S_{3p} & S_{2p} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ S_{np} & S_{(n-1)p} & \cdots & S_p \end{vmatrix} ,$$

or, alternatively,

$$\begin{aligned} &S_p + A_1 = 0 \\ &S_{2p} + A_1 S_p + 2A_2 = 0 \\ &S_{3p} + A_1 S_{2p} + A_2 S_p + 3A_3 = 0 \\ &\vdots \\ &S_{np} + A_1 S_{(n-1)p} + A_2 S_{(n-2)p} + \cdots + A_{n-1} S_p + nA_n = 0. \end{aligned} \tag{2.28}$$

$S_p, S_{2p}, \ldots, S_{np}$ could be obtained from (2.23) or (2.25) and $A_1, A_2, \ldots, A_n$ from (2.27) or (2.28). Obviously, the coefficients of $P_{p+1}(x)$, $P_{p+2}(x)$, etc., can be found similarly.

CHAPTER III

ROOT DETERMINATION

The identification of the types of roots--real, imaginary, equal moduli, etc.--as well as their numerical determination, is considerably simplified by having a sequence of equations whose roots are consecutive powers of the roots of $P(x) = 0$. For the equation whose roots are the $m^{\underline{th}}$ powers of the roots of $P(x) = 0$, we will use the notation

$$(3.1)\quad P_m(x) = a_o^{(m)} x^n + a_1^{(m)} x^{n-1} + \cdot \cdot \cdot + a_{n-1}^{(m)} x + a_n^{(m)} = 0$$

Case I. Roots all real with unequal absolute values

As with root-squaring, a power, p, of the roots can be found so that the ratio of the $p^{\underline{th}}$ power of any root to the $p^{\underline{th}}$ power of the next numerically larger root is negligible compared to one in absolute value:

$$(3.2)\qquad \left|\frac{x_{k+1}^p}{x_k^p}\right| << 1,\ k = 1, 2, 3, \cdot \cdot \cdot , n-1$$

where

$$(3.3)\qquad |x_1| > |x_2| > \cdot \cdot \cdot > |x_k| > |x_{k+1}| > \cdot \cdot \cdot > |x_n| .$$

Again using the Newton elementary symmetric functions on $P_p(x)$ gives

$$(3.4)\quad \frac{a_1^{(p)}}{a_o^p} = -(x_1^p + x_2^p + \cdot \cdot \cdot + x_n^p) = -x_1^p \left(1 + \frac{x_2^p}{x_1^p} + \frac{x_3^p}{x_1^p} + \cdot \cdot \cdot + \frac{x_n^p}{x_1^p}\right) \doteq -x_1^p$$

$$\frac{a_2^{(p)}}{a_o^p} = x_1^p x_2^p + x_1^p x_3^p + \cdot \cdot \cdot + x_1^p x_n^p + x_2^p x_3^p + \cdot \cdot \cdot + x_{n-1}^p x_n^p$$

$$= x_1^p x_2^p \left(1 + \frac{x_3^p}{x_2^p} + \cdot \cdot \cdot + \frac{x_n^p}{x_2^p} + \frac{x_3^p}{x_1^p} + \cdot \cdot \cdot + \frac{x_{n-1}^p x_n^p}{x_1^p x_2^p}\right)$$

$$\doteq x_1^p\, x_2^p$$

$$\vdots$$

$$\frac{a_k^{(p)}}{a_0^p} = (-1)^k[x_1^p x_2^p \; . \; . \; . \; x_k^p + x_1^p x_2^p \; . \; . \; . \; x_{k-1}^p x_{k+1}^p + \; . \; . \; .$$

$$+ x_{n-k}^p x_{n-k+1}^p \; . \; . \; . \; x_n^p]$$

$$= (-1)^k x_1^p x_2^p \; . \; . \; . \; x_k^p \left(1 + \frac{x_{k+1}^p}{x_k^p} + \; . \; . \; . \; . + \frac{x_{n-k}^p \; . \; . \; . \; x_n^p}{x_1^p \; . \; . \; . \; x_k^p} \right)$$

$$\doteq (-1)^k x_1^p x_2^p \; . \; . \; . \; x_k^p$$

$$\frac{a_{k+1}^{(p)}}{a_0^p} = (-1)^{k+1} x_1^p x_2^p \; . \; . \; . \; x_k^p x_{k+1}^p \left(1 + \frac{x_{k+2}^p}{x_{k+1}^p} + \; . \; . \; . \; . + \frac{x_{n-k-1}^p \; . \; . \; . \; x_n^p}{x_1^p \; . \; . \; . \; x_{k+1}^p} \right)$$

$$\doteq (-1)^{k+1} x_1^p \; . \; . \; . \; x_{k+1}^p$$

$$\vdots$$

$$\frac{a_n^{(p)}}{a_0^p} = (-1)^n x_1^p x_2^p \; . \; . \; . \; x_n^p \; .$$

Also,

$$(3.5) \qquad \frac{a_k^{(p+1)}}{a_0^{p+1}} = (-1)^k (x_1^{p+1} x_2^{p+1} \; . \; . \; . \; x_k^{p+1} + \; . \; . \; . \; . + x_{n-k}^{p+1} x_{n-k+1}^{p+1} \; . \; . \; . \; x_n^{p+1}$$

$$= (-1)^k x_1^{p+1} x_2^{p+1} \; . \; . \; . \; x_k^{p+1} \left(1 + \frac{x_{k+1}^{p+1}}{x_k^{p+1}} + \; . \; . \; . \; . + \frac{x_{n-k}^{p+1} \; . \; . \; . \; x_n^{p+1}}{x_1^{p+1} \; . \; . \; . \; x_k^{p+1}} \right)$$

$$\doteq (-1)^k x_1^{p+1} x_2^{p+1} \; . \; . \; . \; x_k^{p+1}$$

for $k = 1, 2, . . . , n$. Clearly, then,

$$(3.6)\quad \frac{a_{k+1}^{(p+1)}\,a_k^{(p)}}{a_k^{(p+1)}\,a_{k+1}^{(p)}} \doteq x_{k+1}, \quad k = 0, 1, 2, \cdots, n-1;$$

and the roots are found with no ambiguity of sign.

From the relations (3.4), (3.5) and an analogous one for $p + 2$, we note

$$(3.7)\quad \frac{a_k^{(p+1)}}{a_k^{(p)}} \doteq a_o x_1 x_2 \cdots x_k \doteq \frac{a_k^{(p+2)}}{a_k^{(p+1)}} .$$

Thus, the power, p, for (3.2) to hold is reached when the corresponding coefficients in three consecutive root-powerings are, within the limits of precision required, in geometric progression.

Case II. Real roots of equal absolute value

Since multiple roots may be determined by other means (g.c.d. of $P(x)$ and $P'(x)$), we consider only the case

$$(3.8)\quad |x_{k-1}| > x_k = -x_{k+1} > |x_{k+2}| .$$

Again, p can be found such that

$$(3.9)\quad \frac{x_k^p}{|x_{k-1}^p|} << 1 \text{ and } \frac{|x_{k+2}^p|}{x_k^p} << 1.$$

The symmetric functions involved are

$$(3.10)\quad \frac{a_{k-1}^{(p)}}{a_o^p} = (-1)^{k-1}(x_1^p \cdots x_{k-1}^p + x_1^p \cdots x_{k-2}^p x_k^p + \cdots + x_{n-k+1}^p \cdots x_n^p)$$

$$= (-1)^{k-1} x_1^p \cdots x_{k-1}^p \left(1 + \frac{x_k^p}{x_{k-1}^p} + \cdots + \frac{x_{n-k+1}^p \cdots x_n^p}{x_1^p \cdots x_{k-1}^p}\right)$$

$$\doteq (-1)^{k-1}\, x_1^p \cdots x_{k-1}^p \ ,$$

$$\frac{a_k^{(p)}}{a_o^p} = (-1)^k \,[x_1^p \cdots x_{k-1}^p\,(x_k^p + x_{k+1}^p) + x_1^p \cdots x_{k-1}^p x_{k+2}^p + \cdots + x_{n-k-1}^p \cdots x_n^p]$$

$$= (-1)^k 2x_1^p \cdots x_{k-1}^p\, x_k^p \left(1 + \frac{x_{k+2}^p}{2x_k^p} + \cdots + \frac{x_{n-k-1}^p \cdots x_n^p}{2x_1^p \cdots x_k^p}\right)$$

$$\doteq (-1)^k\, 2x_1^p \cdots x_{k-1}^p x_k^p \text{, if p is even,}$$

$$\frac{a_k^{(p)}}{a_o^p} = (-1)^k\, x_1^p \cdots x_{k-1}^p\, x_{k+2}^p \left(1 + \cdots + \frac{x_{n-k-1}^p \cdots x_n^p}{x_1^p \cdots x_{k-1}^p x_{k+2}^p}\right)$$

$$\doteq (-1)^k x_1^p \cdots x_{k-1}^p x_{k+2}^p \text{, if p is odd,}$$

$$\frac{a_{k+1}^{(p)}}{a_o^p} = (-1)^{k+1}\,(x_1^p \cdots x_{k-1}^p x_k^p x_{k+1}^p + x_1^p \cdots x_k^p x_{k+2}^p + \cdots + x_{n-k-1}^p \cdots x_n^p)$$

$$= (-1)^{k+1} x_1^p \cdots x_{k-1}^p\, (-1)^p x_k^{2p} \left(1 + \frac{x_{k+2}^p}{x_{k+1}^p} + \cdots + \frac{x_{n-k-1}^p \cdots x_n^p}{x_1^p \cdots x_{k-1}^p x_k^p x_{k+1}^p}\right)$$

$$\doteq (-1)^{k+1+p} x_1^p \cdots x_{k-1}^p x_k^{2p}$$

with similar expressions for p + 1 and p + 2. Since successive powers alternate in parity, $a_k^{(m)}$ will alternate between $(-1)^k 2a_o^m x_1^m \cdots x_k^m$ and $(-1)^k a_o^m x_1^m \cdots x_{k-1}^m x_{k+2}^m$ for m = p, p + 1, p + 2, etc. Because of (3.9), the former for even order-powering will tend to increase by numerically larger factors than the latter odd order-powering. That is, this case will be recognized by the irregular behavior of $a_k^{(m)}$ increasing by numerically larger factors for even order-powering than for odd order-powering.

Immediately from (3.10),

$$(3.11)\quad \frac{a_{k+1}^{(p+1)}}{a_{k-1}^{(p+1)}} \doteq x_k^{2p+2},\ \frac{a_{k+1}^{(p)}}{a_{k-1}^{(p)}} \doteq x_k^{2p},$$

so that

$$(3.12)\quad x_k^2 \doteq \frac{a_{k+1}^{(p+1)}}{a_{k-1}^{(p+1)}} \cdot \frac{a_{k-1}^{(p)}}{a_{k+1}^{(p)}} .$$

Case III. Complex conjugate roots

Suppose

$$(3.13)\ x_k = \bar{x}_{k+1} = \rho_k\ e^{i\phi k} \text{ and } \left|x_{k-1}\right| > \rho_k > \left|x_{k+2}\right| .$$

Also, that p is such that

$$(3.14)\ \frac{\rho_k^{p-1}}{\left|x_{k-1}^{p-1}\right|} << 1 \quad \text{and} \quad \frac{\left|x_{k+2}^{p-1}\right|}{\rho_k^{p-1}} << 1 .$$

The pertinent symmetric functions become

$$(3.15)\quad \frac{a_{k-1}^{(p)}}{a_o^p} = (-1)^{k-1}\ (x_1^p x_2^p \cdot\cdot\cdot x_{k-1}^p + x_1^p \cdot\cdot\cdot x_{k-2}^p x_k^p + \cdot\cdot\cdot$$

$$+ x_{n-k+1}^p \cdot\cdot\cdot x_n^p)$$

$$= (-1)^{k-1} x_1^p \cdot\cdot\cdot x_{k-1}^p \left(1 + \frac{x_k^p}{x_{k-1}^p} + \cdot\cdot\cdot + \frac{x_{n-k+1}^p \cdot\cdot\cdot x_n^p}{x_1^p \cdot\cdot\cdot x_{k-1}^p}\right)$$

$$\doteq (-1)^{k-1}\ x_1^p \cdot\cdot\cdot x_{k-1}^p \text{, since } \left|\frac{x_k^p}{x_{k-1}^p}\right| = \frac{\rho_k^p}{\left|x_{k-1}^p\right|} << 1, \text{ etc.}$$

$$\frac{a_k^{(p)}}{a_o^p} = (-1)^k \Big[x_1^p \cdots x_{k-1}^p (x_k^p + x_{k+1}^p) + x_1^p \cdots x_{k-1}^p x_{k+2}^p + \cdots + x_{n-k}^p \cdots x_n^p\Big]$$

$$= (-1)^k x_1^p \cdots x_{k-1}^p \rho_k^p \, 2 \cos p\,\phi \left(1 + \frac{x_{k+2}^p}{\rho_k^p \, 2 \cos p\,\phi} + \cdots + \frac{x_{n-k}^p \cdots x_n^p}{x_1^p \cdots x_{k-1}^p \rho_k^p \, 2 \cos p\,\phi}\right)$$

$$\doteq (-1)^k x_1^p \cdots x_{k-1}^p \rho_k^p \, 2 \cos p\,\phi \text{ , since } \frac{\left|x_{k+2}^p\right|}{\rho_k^p} < < 1, \text{ etc.}$$

$$\frac{a_{k+1}^{(p)}}{a_o^p} = (-1)^{k+1}(x_1^p \cdots x_{k-1}^p \rho_k^{2p} + \cdots + x_{n-k-1}^p \cdots x_n^p)$$

$$\doteq (-1)^{k+1} x_1^p \cdots x_{k-1}^p \rho_k^{2p}.$$

From these relations,

$$(3.16) \quad \frac{a_k^{(p)}}{a_{k-1}^{(p)}} \doteq -2\rho_k^p \cos p\,\phi ,$$

$$\frac{a_{k+1}^{(p)}}{a_{k-1}^{(p)}} \doteq \rho_k^{2p} .$$

Similarly,

$$(3.17) \quad \frac{a_k^{(p-1)}}{a_{k-1}^{(p-1)}} \doteq -2\rho_k^{p-1} \cos (p-1)\,\phi , \quad \frac{a_{k+1}^{(p-1)}}{a_{k-1}^{(p-1)}} \doteq \rho_k^{2(p-1)} ,$$

$$(3.18) \quad \frac{a_k^{(p+1)}}{a_{k-1}^{(p+1)}} \doteq -2\rho_k^{p+1} \cos (p+1)\,\phi , \quad \frac{a_{k+1}^{(p+1)}}{a_{k-1}^{(p+1)}} \doteq \rho_k^{2(p+1)}.$$

Thus, immediately,

$$(3.19)\quad \frac{a_{k+1}^{(p)}}{a_{k-1}^{(p)}} \cdot \frac{a_{k-1}^{(p-1)}}{a_{k+1}^{(p-1)}} \doteq \frac{a_{k+1}^{(p+1)}}{a_{k-1}^{(p+1)}} \cdot \frac{a_{k-1}^{(p)}}{a_{k+1}^{(p)}} \doteq \rho_k^2 .$$

From (3.18) and (3.19)

$$-\frac{2\rho_k^{p+1}\cos(p\phi+\phi)}{\rho_k^2} = -2\rho^{p-1}\cos(p\phi+\phi) \doteq \frac{a_k^{(p+1)}}{a_{k-1}^{(p+1)}} \cdot \frac{a_{k-1}^{(p+1)}}{a_{k+1}^{(p+1)}} \cdot \frac{a_{k+1}^{(p)}}{a_{k-1}^{(p)}}$$

$$(3.20)\qquad \doteq \frac{a_k^{(p+1)}a_{k+1}^{(p)}}{a_{k+1}^{(p+1)}a_{k-1}^{(p)}} .$$

Now, using the relation $\cos(p\phi-\phi)+\cos(p\phi+\phi) = 2\cos p\phi\cos\phi$ together with (3.17) and (3.20), we find that

$$(3.21)\quad -4\rho_k^{p-1}\cos p\phi\cos\phi \doteq \frac{a_k^{(p-1)}}{a_{k-1}^{(p-1)}} + \frac{a_k^{(p+1)}a_{k+1}^{(p)}}{a_{k+1}^{(p+1)}a_{k-1}^{(p)}} .$$

Finally, using (3.16), (3.19) and (3.21), we obtain

$$\cos\phi \doteq \frac{\dfrac{a_k^{(p-1)}}{a_{k-1}^{(p-1)}} + \dfrac{a_k^{(p+1)}a_{k+1}^{(p)}}{a_{k+1}^{(p+1)}a_{k-1}^{(p)}}}{2\dfrac{a_k^{(p)}}{a_{k-1}^{(p)}}\left[\dfrac{a_{k+1}^{(p+1)}}{a_{k-1}^{(p+1)}}\cdot\dfrac{a_{k-1}^{(p)}}{a_{k+1}^{(p)}}\right]^{\frac{1}{2}}} ,$$

$$(3.22)\quad \doteq \frac{1}{2}\left[\frac{a_k^{(p-1)}a_{k-1}^{(p)}}{a_{k-1}^{(p-1)}a_k^{(p)}} + \frac{a_k^{(p+1)}a_{k+1}^{(p)}}{a_{k+1}^{(p+1)}a_k^{(p)}}\right]\left[\frac{a_{k+1}^{(p+1)}a_{k-1}^{(p)}}{a_{k-1}^{(p+1)}a_{k+1}^{(p)}}\right]^{-\frac{1}{2}}$$

Consideration of (3.15) and the corresponding expressions for p-1 and p+1 indicate that the coefficients $a_{k-1}^{(m)}$ and $a_{k+1}^{(m)}$ will show behavior like case I for m = large p; i.e., $a_{k-1}^{(p-1)}$, $a_{k-1}^{(p)}$, $a_{k-1}^{(p+1)}$ and $a_{k+1}^{(p-1)}$, $a_{k+1}^{(p)}$, $a_{k+1}^{(p+1)}$ will become geometric progressions. However, $a_k^{(m)}$ will show irregular behavior, both in sign and magnitude, due to the factor cos $m\phi$. This irregularity, coupled with the regularities of the adjacent coefficients, provides the criterion for the recognition of the pair of complex conjugate roots.

Case IV. Roots with the same modulus

Suppose several, ν, roots have the same modulus, ρ_k, but different amplitudes, ϕ_{k+r}. That is,

$$(3.23)\quad x_{k+r} = \rho_k e^{i\phi_{k+r}}, \quad r = 0, 1, 2, \ldots, \nu-1$$

and

$$(3.24)\quad |x_{k-1}| > \rho_k > |x_{k+\nu}|.$$

The dominant parts of the pertinent symmetric functions for large p are, then,

$$(3.25)\quad \frac{a_{k-1}^{(p)}}{a_o^p} \doteq (-1)^{k-1} x_1^p x_2^p \ldots x_{k-1}^p$$

$$\frac{a_k^{(p)}}{a_o^p} \doteq (-1)^k x_1^p \ldots x_{k-1}^p \rho_k^p \sum_{r=0}^{\nu-1} \cos p\,\phi_{k+r}$$

$$\frac{a_{k+1}^{(p)}}{a_o^p} \doteq (-1)^{k+1} x_1^p \ldots x_{k-1}^p \rho_k^{2p} \left(\sum_{\substack{r_1=1 \\ r_1 > r_2}}^{\nu-1} \sum_{r_2=0}^{\nu-2} \cos p\,\phi_{k+r_1} \cos p\,\phi_{k+r_2} + \frac{1}{2} \sum_{r=0}^{\nu-1} \sin^2 p\,\phi_{k+r} \right)$$

$$\vdots$$

$$\frac{a_{k+\nu-2}^{(p)}}{a_o^p} \doteq (-1)^{k+\nu-2} x_1^p \ldots x_{k-1}^p \rho_k^{(\nu-1)p} \sum_{s=0}^{\nu-1} \frac{\prod_{r=0}^{\nu-1} e^{i\phi_{k+r}}}{e^{i\phi_{k+s}}}$$

$$\doteq (-1)^{k+\nu-2} x_1^p \cdots x_{k-1}^p \rho_k^{(\nu-1)p} \left(-\sum_{r=0}^{\nu-1} \cos p\,\phi_{k+r} \right),$$

if $-\rho_k$ is a root and p is odd,

$$\doteq (-1)^{k+\nu-2} x_1^p \cdots x_{k-1}^p \rho_k^{(\nu-1)p} \sum_{r=0}^{\nu-1} \cos p\,\phi_{k+r}, \text{ otherwise,}$$

$$\frac{a_{k+\nu-1}^{(p)}}{a_0^p} \doteq (-1)^{k+\nu} x_1^p \cdots x_{k-1}^p \rho_k^{\nu p}, \text{ if } -\rho_k \text{ is a root and } p \text{ is odd,}$$

$$\doteq (-1)^{k+\nu-1} x_1^p \cdots x_{k-1}^p \rho_k^{\nu p}, \text{ otherwise.}$$

From these relations, it is clear that $a_{k-1}^{(p)}$ and $a_{k+\nu-1}^{(p)}$ will show regular geometric progression behavior for large consecutive values of p, while all the coefficients between will be irregular because of the trigonometric factors involved.

For small multiplicities, ν, the trigonometric relations could be used to determine the ϕ_{k+r} by finding $\cos\,(p-1)\phi_{k+r}$ and $\cos\,(p+1)\phi_{k+r}$ and, hence, $\cos\,\phi_{k+r}$. For larger multiplicities, the resultant procedure mentioned in Chapter I might be employed; ρ_k, of course, would be found from

$$\rho_k^{\nu} \doteq \left| \frac{a_{k+\nu-1}^{(p+1)}}{a_{k-1}^{(p+1)}} \; \frac{a_{k-1}^{(p)}}{a_{k+\nu-1}^{(p)}} \right| . \tag{3.26}$$

CHAPTER IV

SUMMARY AND CONCLUSIONS

Classical root-squaring and more recent root-cubing are found to be special cases of more general root-powering as expressed in the fundamental theorem. The coefficients of the powered equations are derived formally in three different ways; as convolution products, by means of circulants and in terms of Newton's symmetric functions. A fourth procedure, powering of the companion matrix, is also discussed.

Although the convolution representation of the coefficients is formally concise, its implementation for high powers in numerical examples becomes quite tedious. Also, it lacks the desirable property of yielding consecutive powers in any iterative sense. That is, each new power requires a completely separate and independent treatment.

For practical utilization of the circulant method for high-degree equations, computer programming of both the recurrence relations and the initial determinant calculations would seem to be necessary. However, once started, the continuing calculation of coefficients for successively higher powers is simple and iterative.

The Newton symmetric function procedure offers no special difficulties for a desk calculator, except, possibly, in tabulation of S-values for later use. It appears to this writer that this method might well be further exploited.

Matrix-powering offers particularly efficient and easy methods of powering, but entails the problem of determining the powered characteristic equations. Further investigations of these powered matrices with respect to their characteristic equations or their characteristic roots might lead to useful results. The traces of the successively powered matrices offer another way of finding the Newton symmetric functions needed for that method. That is,

$$(4.1)\quad \mathrm{tr}(A) = S_1 , \quad \mathrm{tr}(A^2) = S_2 , \ . \ . \ . \ , \mathrm{tr}(A^k) = S_k .$$

These S's can then be used in (2.27) or (2.28) to find the coefficients of the powered equations.

Essentially, from the consecutively powered equations, the actual roots are estimated. Besides the obvious advantage over root-squaring of being able to determine the unpowered complex roots by elementary methods, there is the possibility of finding successive approximations to the roots. That is, the roots can be estimated at each successive powering and their convergence used as a criterion for their precision and for the termination of the powering process.

APPENDIX

NUMERICAL EXAMPLES

The roots of the equation,

$$(A.1)\quad (2x+1)(x+2)(x-5) = 2x^3 - 5x^2 - 23x - 10 = 0$$

are powered by the usual root-squaring method and by the four methods developed in Chapter II. Using these results, the roots are approximated by the methods of Chapter III.

Root-squaring method

(A.2)

Power				
1	2	-5	-23	-10
2	4	117	429	100
4	16	10257	160641	1(4)
8	256	10007(4)	25600(6)	1(8)
16	65536	10001(12)	65534(16)	1(16)

The notation, (N), indicates the power-of-ten factor. The equation whose roots are $-(-\frac{1}{2})^{16}$, $-(-2)^{16}$ and $-(5)^{16}$ is, then,

$$(A.3)\quad 65536x^3 + 10001(12)x^2 + 65534(16)x + 1(16) = 0\,,$$

or the equation whose roots are $(-\frac{1}{2})^{16}$, $(-2)^{16}$ and $(5)^{16}$ is

$$(A.4)\quad 65536x^3 - 10001(12)x^2 + 65534(16)x - 1(16) = 0.$$

Convolution

We find the coefficients of the equation whose roots are the fourth powers of the roots of (A.1) using the relations

$\omega_4 = i$, $\omega_4^2 = -1$, $\omega_4^3 = -i$ and performing the convolution products in the order $\left\{A_k\right\} = \left\{\omega_4^k a_k\right\} * \left\{\omega^{2k} a_k\right\} * \left\{\omega^{3k} a_k\right\} * \left\{a_k\right\}$.

(A.5)

A_0	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
2	-5i	23	10i									
2	5	-23	10									
4	10(1-i)	25i	135(1+i)	-529	230(1-i)	100i						
2	5i	23	-10i									
8	20	142	625	-1833	3315	-9667	4790	-2300	1000			
2	-5	-23	-10									
16	0	0	0	-10257	0	0	0	160641	0	0	0	-1(4)

The desired equation is

$$(A.6)\quad 16x^3 - 10257x^2 + 160641x - 10000 = 0\ .$$

Circulant

The six by six circulant, from which all powered equations for this case are derived, is

$$(A.7)\quad \begin{pmatrix} 2x & -5x & -23x & -10 & 0 & 0 \\ 0 & 2x & -5x & -23 & -10 & 0 \\ 0 & 0 & 2x & -5 & -23 & -10 \\ -10 & 0 & 0 & 2 & -5 & -23 \\ -23 & -10 & 0 & 0 & 2 & -5 \\ -5 & -23 & -10 & 0 & 0 & 2 \end{pmatrix}\ .$$

The half-sequences and their iteration formulae are (A.8)

$$\begin{array}{llll} p = r+1 & & p = r & \\ \left|;\, 4,\, 5,\, 6\right| & = & a_0\ \left|;\, 4,\, 5,\, 6\right| & = 2\ \left|;\, 4,\, 5,\, 6\right| \\ \left|;\, 3,\, 5,\, 6\right| & = & 10\ \left|;\, 1,\, 5,\, 6\right| & \end{array}$$

$|;3,4,6| = 10|;1,4,6|$
$|;3,4,5| = 10|;1,4,5|$
$|;2,5,6| = 2|;3,5,6| + 23|;1,5,6|$
$|;2,4,6| = -2|;2,5,6|$
$|;2,4,5| = -2|;2,4,6|$
$|;1,5,6| = 2|;2,5,6| + 5|;1,5,6|$
$|;1,4,6| = -2|;1,5,6|$
$|;1,4,5| \quad -2|;1,4,6|$

$|;2,3,6| = -10|;1,3,6|$
$|;2,3,5| = -10|;1,3,5|$
$|;2,3,4| = -10|;1,3,4|$
$|;1,3,6| = -10|;1,2,6|$
$|;1,3,5| = -10|;1,2,5|$
$|;1,3,4| = -10|;1,2,4|$
$|;1,2,6| = -23|;1,2,6| + 10|;1,2,5|$
$|;1,2,5| = -5|;1,2,6| + 10|;1,2,4|$
$|;1,2,4| = 2|;1,2,6|$

$|;1,2,3| = 10|;1,2,3|$.

Listing, in the same order as (A.8), the values of these with their cofactors gives:

Cofactor	p=6	p=7	p=8 ...	p=14	p=15	p=16
$8x_3$	8	16	32	2048	4096	8192
$20x^2$	-710	-6250	-65910	-64939(6)	-64934(7)	-64936(8)
$-92x^2$	100	1420	12500	12985(6)	12987(7)	12986(7)
$40x^2$	-40	-200	-2840	-25984(5)	-25970(6)	-25974(7)
$142x^2$	-1733	-15795	-164093	-16234(7)	-16234(8)	-16234(9)
$-270x^2$	270	3466	31590	32464(6)	32469(6)	32467(8)
$100x^2$	-92	-540	-6932	-64952(5)	-64928(6)	-65938(7)
$625x^2$	-625	-6591	-64545	-64934(6)	-64936(7)	-64936(8)
$-1733x^2$	142	1250	13182	12987(6)	129867(6)	12987(8)
$710x^2$	-20	-284	-2500	-25970(5)	-25974(6)	-25973(7)
$958x$	-2300	47900	-966700	-62060(9)	12412(11)	-24824(12)
$-460x$	-500	13500	-285500	-18618(9)	37236(10)	-74472(11)
$200x$	200	-4600	95800	62060(8)	-12412(10)	24824(11)
$2855x$	-4790	96670	-1937910	-12412(10)	24824(11)	-49648(12)
$-1350x$	-1350	28550	-579150	-37236(9)	74472(10)	-14894(12)
$500x$	460	-9580	193340	12412(9)	-24824(10)	49648(11)
$-9667x$	-9667	193791	-3878043	-24824(10)	49648(11)	-99296(12)
$4790x$	-2855	57915	-1162295	-74472(9)	14894(11)	-29789(12)
$-2300x$	958	-19334	387582	24824(9)	-49648(10)	99296(11)
1000	-1000	-1(4)	-1(5)	-1(11)	-1(12)	-1(13)

The sum of the binary products of a term in the cofactor column times the corresponding term under an indicated p gives the equation whose roots are the $p^{\underline{th}}$ powers of (A.1). For p = 14, 15, and 16 these are, respectively:

$$
\begin{aligned}
&16384x^3 - 99999(9)x^2 + 16384(14)x - 1(14) = 0 \\
\text{(A.9)}\qquad &32768x^3 - 99999(10)x^2 - 32768(15)x - 1(15) = 0 \\
&65536x^3 - 10001(12)x^2 + 65534(16)x - 1(16) = 0 \ .
\end{aligned}
$$

Matrix-powering

We first divide (A.1) through by 2 to obtain

$$\text{(A.10)}\qquad x^3 - 2.5x^2 - 11.5x - 5 = 0 \ .$$

For ease in tabulating, the order of the rows of the companion matrix of (A.10) is reversed and the new rows for each higher power added at the bottom.

Power			
	0	1	0
	1	0	0
1	2.5	11.5	5
2	17.75	33.75	12.5
3	78.125	216.625	88.75
4	411.9375	987.1875	390.625

For the equation whose roots are the $4^{\underline{th}}$ powers of the roots of (A.10), we transform the matrix

$$\text{(A.11)}\qquad A^4 = \begin{pmatrix} 411.9375 & 987.1875 & 390.625 \\ 78.125 & 216.625 & 88.75 \\ 17.75 & 33.75 & 12.5 \end{pmatrix}$$

by the method of Danilevský [8, pp. 166-176].

	411.9375	987.1875	390.625
	78.125	216.625	88.75
	17.75	33.75	12.5
	-.5259259		-.3703704
17.75	- 107.25	29.25	25
33.75	- 35.8037	6.418518	8.518519
12.5	0	1	0
	-3112.062	748.3125	731.25
		.2404555	.2349728
-3112.062	.03446268	3.461149	-.2008316
748.3125	1	0	0
731.25	0	1	0
	641.063	-10040.1	625

The desired equation is, then,

$$x^3 - 641.063x^2 + 10040.1x - 625 = 0 . \qquad (A.12)$$

This agrees to five significant figures to the exact coefficients from root-squaring (A.2).

Symmetric functions

Here, again, we use the equation in the form (A.10) and find the equation whose roots are the 4th powers of (A.10). For this, we need the values of S_k up to $S_{np} = S_{12}$; they are calculated here by (2.25).

$$S_1 = -a_1 = 2.5$$

$$S_2 = -a_1S_1 - 2a_2 = 2.5 \times 2.5 + 23 = 29.25$$

$$S_3 = -a_1S_2 - a_2S_1 - 3a_3 = 116.875$$

$$S_4 = -a_1S_3 - a_2S_2 - a_3S_1 = 641.0625$$

$$S_5 = -a_1S_4 - a_2S_3 - a_3S_2 = 3092.969 \qquad (A.13)$$

$$S_6 = 15689.02$$

$$S_7 = 77996.99$$

$$S_8 = 390881$$

$$S_9 = 1952613$$

$$S_{10} = 9766649$$

$$S_{11} = 48826080$$

$$S_{12} = 244144700$$

The coefficients, A_1, A_2, A_3 of the powered equation are found from these by (2.28);

$$A_1 = -S_4 = -641.0625$$

$$A_2 = -\tfrac{1}{2}(S_8 + A_1S_4) = 10040.1 \qquad \text{(A.14)}$$

$$A_3 = -\frac{1}{3}(S_{12} + A_1S_8 + A_2S_4) = -625.13\ .$$

Determination of roots

From the last two equations of (A.9) for the 15th and 16th powers, we calculate the roots by (3.6).

$$x_1 \doteq \frac{a_1^{(16)}\ a_o^{(15)}}{a_o^{(16)}\ a_1^{(15)}} = \frac{10001(12) \text{ x } 32768}{65536 \text{ x } 99999(10)} = 5.003$$

$$x_2 \doteq \frac{a_2^{(16)}\ a_1^{(15)}}{a_1^{(16)}\ a_2^{(15)}} = -\frac{65534(16) \text{ x } 99999(10)}{10001(12) \text{ x } 32768(15)} = -\ 1.9997$$

$$x_3 \doteq \frac{a_3^{(16)}\ a_2^{(15)}}{a_2^{(16)}\ a_3^{(15)}} = -\ \frac{1(16) \text{ x } 32768(15)}{65534(16) \text{ x } 1(15)} = -\ .50002\ .$$

Similarly, using the first two equations of (A.9) for the 14th and 15th powers,

$$x_1 \doteq \frac{99999(10) \text{ x } 16384}{32768 \text{ x } 99999(9)} = 5.000$$

$$x_2 \doteq -\ \frac{32768(15) \text{ x } 99999(9)}{99999(10) \text{ x } 16384(14)} = -\ 2.000$$

$$x_3 \doteq -\ \frac{1(15) \text{ x } 16384(14)}{32768(15) \text{ x } 1(14)} = -\ .5000\ ,$$

and we have agreement in all cases to at least three significant figures.

For an example with repeated and complex roots we use

$$x^4 - .6x^3 - 13.2x^2 - 36x - 40 = 0 \tag{A.15}$$

whose roots are known to be 5, -2, $\frac{1}{5}(-6 \pm 8i)$. The equations, found by the use of Newton's symmetric functions, for the 9th, 10th, 11th and 12th powers of the roots are, respectively,

$$\begin{aligned}
&x^4 - 1953096x^3 - 557785(2)x^2 - 283(8)x - 262144(9) = 0\\
&x^4 - 9764625x^3 - 9770955(3)x^2 + 1(13)x + 1048576(10) = 0\\
&x^4 - 48829(3)x^3 + 42817(6)x^2 + 4/3\ (14)x - 4194304(11) = 0\\
&x^4 - 2441458(2)x^3 + 12632(8)x^2 + .333(16) + 16777216(12) = 0
\end{aligned} \tag{A.16}$$

Immediately, the second coefficient is seen to form a geometric progression, the next two are irregular and the third is regular as always. Thus, the largest root is given by

$$x_1 = \frac{-48829(3)}{-9764625} = 5.0006$$

$$x_1 = \frac{-2441458(2)}{-48829(3)} = 5.00002$$

as two successive approximations. The remaining three roots must have equal moduli, this given by

$$\rho^3 = \frac{16777216(12) \text{ x } 48829(3)}{244158(2) \text{ x } 4194304(11)} = 7.99997$$

or $\rho \doteq 2$. Examination of the signs of the second and last coefficients indicate the remaining real root to be $- \sqrt[3]{7.99997} \doteq -2$. The amplitudes of the remaining two conjugate imaginary roots are estimated by use of trigonometric relations.

$$(A.17) \quad \frac{a_2^{(m)}}{a_1^{(m)}} \doteq -\rho^m \left[(-1)^m + 2 \cos m \phi \right]$$

for $m = p-1 = 10$, $p = 11$, $p+1 = 12$. $a_3^{(m)}$ is not used in this case, because of its loss of significant figures due to round-off error. With some elementary manipulation, we obtain

$$(A.18) \quad \cos \phi = \frac{\dfrac{a_2^{(12)}}{a_1^{(12)}} + \dfrac{\rho^2 a_2^{(10)}}{a_1^{(10)}} + 2\rho^{12}}{2\left(\dfrac{\rho a_2^{(11)}}{a_1^{(11)}} - \rho^{12} \right)} .$$

Using the data from (A.16) and $\rho = 2$ in (A.18) gives $\cos \phi = -.60007$.

	Estimated value	Actual value
x_1	5.00002	5
x_2	-2.00000	-2
ρ	2.00000	2
$\cos \phi$	- .60007	-.6

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BIOGRAPHICAL SKETCH

Francis C. Hatfield was born April 20, 1919, at Savanna, Illinois. He attended public schools in Illinois and Minnesota, graduating from Elkton High School, Elkton, Minnesota in 1937. His undergraduate college work was interrupted by military service from July, 1941 to December, 1945, during which time he served first as an enlisted man and later as a commissioned officer in the United States Army Field Artillery. In June, 1946 he received the degree of Bachelor of Science in Education from Winona State College. The Master of Science Degree was awarded to him in June, 1950 from the University of Minnesota. After several additional summers work at the University of Iowa and the University of Minnesota, he began his doctoral program at the University of Florida in September, 1958. He studied under a Danforth Teacher Grant for the year 1958-1959 and from January, 1963 to the present time.

Mr. Hatfield has taught for a total of eleven and one-half years: two years at Bemidji State College, one year at Blackburn College, eight years at Mankato State College and one-half year at the University of Florida. He is on a leave of absence from Mankato State College for this academic year.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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